

Strategy-Proof Voting Schemes with Continuous Preferences*

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Received July 12, 1988 / Accepted April 6, 1989

Abstract. We prove that all nondictatorial voting schemes whose range has more than two alternatives will be manipulable, when their domain is restricted to the set of all continuous preferences over alternatives. Our result neither implies nor is implied by the original Gibbard-Satterthwaite theorem, except if the number of alternatives is finite, when they coincide. A new, direct line of reasoning is used in the proof. It is presented in an introductory section, which may be useful in classroom situations.

1. Introduction

For a society with n individuals facing a set \mathcal{A} of alternatives, a voting scheme is a function which determines what alternative to choose on the basis of the preferences of individuals. The Gibbard-Satterthwaite theorem (Gibbard 1973; Satterthwaite 1975) establishes that (except for minor trivial cases), all voting schemes are manipulable – that is, there are always situations under which some individual's best strategy is not to reveal his true preferences. The theorem has justly become a classic, because it expresses in simple and categorical terms the need for a systematic analysis of the strategic properties of mechanisms for collective decision making. The theorem's clear-cut conclusion is obtained at some costs. One is that it only applies to functions, not to correspondences. Another is that it relies on an assumption of universal domain. That is, all possible combinations of individual preferences over alternatives are considered to be admissible. Although there exists a general understanding that the theorem is robust to alternative formalizations of the basic framework, it is important to examine its validity for specific cases of general interest. In this paper we prove that a result of the Gibbard-Satterthwaite type applies to the case when the universal domain assumption is dropped, and individual preferences are required to be continuous.

* The work of S. Barberà is partially supported by Research Grant PB 86-0613, from the Dirección General de la Investigación Científica y Técnica, Spanish Ministry of Education.

The assumption that all individual preferences are admissible is very natural when the number of alternatives is finite, but not so much when there is an infinity of possible choices for society. And yet, there are many instances where the set of alternatives is infinite. A leading example is the case where alternatives form some compact set \mathcal{A} of \mathbb{R}^n . This case arises in several contexts. When the coordinates of a point in \mathcal{A} are taken to measure the position of a candidate on different issues, each point may be interpreted as a possible political platform. In location theory, points in \mathbb{R}^2 represent different positions within a certain geographical area. When alternatives are points in \mathbb{R}^n , it is often natural to assume that individual preferences will be continuous. More generally, continuity of preferences can be seen as an interesting requirement when the set of alternatives is a metric space. The main result of this paper is a proof that all nondictatorial voting schemes whose range has more than two alternatives will be manipulable, even when we restrict their domain (and thus, the possible means of manipulating it) to the set of continuous preferences.

Although this is by no means a surprising result, proving it requires the adoption of a strategy which differs from that of the best known approaches to the Gibbard-Satterthwaite theorem. Gibbard's (1973) original proof relies on Arrow's impossibility theorem, and it is known that the latter is still valid when preferences are restricted to be continuous, and even under further restrictions over preferences. Yet, we do not use these facts in our proof, since the usual construction of an auxiliary social welfare function, which is essential in Gibbard's approach, involves the usage of preferences which would be discontinuous for most choices of a metric. The same is true for the more direct proof in Batteau et al. (1981). Other known proofs of the Gibbard-Satterthwaite result use induction arguments on the number of alternatives and do not apply when the number of alternatives is infinite (see Schmeidler and Sonnenschein 1978).

Therefore, a new line of reasoning has to be exploited, which turns out to be very direct. Its bare bones are presented in Sect. 2, both to motivate the general proof and to provide what might be a useful introduction to the theorem in classroom situations. Sect. 3 introduces the general framework and states the theorem. Let us remark that our theorem neither implies nor is implied by the original Gibbard-Satterthwaite result, except for the case when the number of alternatives is finite, in which they coincide. The proof of the theorem is presented in Sects. 4 and 5.

To conclude this Introduction, we should relate our result to previous work on the possibility of the designing strategyproof mechanisms for restricted domains. First of all, there exist results which emphasize the close connection between strategy proofness and Arrow's conditions (Kalai and Muller 1977, for example). These results assume that the rule on which the strategy-proofness requirement is imposed operates over a family of subsets of alternatives and satisfies a rationalizability requirement: roughly speaking, it must be derived from a social welfare function. However, in the absence of such requirement the connection does no longer hold, and one can find strategy proof mechanisms over domains in which no Arrovian Social Welfare functions can be defined. For example, voting by quota is strategy proof when preferences are additively representable, but this is a domain for which any triple of alternatives is free (see Barberà et al. 1988). Another example involving a generalization of single peakedness is provided by Demange (1981).

Since the original Gibbard-Satterthwaite result is valid for functions defined over one single set of alternatives, and requires no rationalizability at all, we think it is important to explore the extensions within this same framework. One good reason for not requiring rationalizability is that such a condition is not natural in economic environments: requiring that economic mechanisms result from maximizing a welfare function is very restrictive indeed!

A second type of results concentrate on domain restrictions which are directly inspired by economic models. An important negative result in this vein is provided by Satterthwaite and Sonnenschein (1982), though their conclusions are obscured by a number of technical assumptions. Our paper explores the implications of strategy proofness alone, for a simple domain restriction. It would be natural to pursue the same line of research for smaller domains, and the technique of proof that we introduce here seems to be promising (see Zhou 1988). But we leave these additional possibilities open.

2. The Gibbard-Satterthwaite Theorem: An Introduction

In this section we present a simple, self-contained proof of the Gibbard-Satterthwaite theorem for the case of linear preferences, two agents and a finite number of alternatives. The proof is self contained, follows the same strategy as our general proof of the main result and uses all of its basic ingredients. [A related but different approach is followed in Barberà (1983).] Remarks at the end of this section will make precise what are the analogies and what are the missing steps.

Let \mathcal{A} be a finite set of *alternatives*. Elements of \mathcal{A} are denoted by x, y, z, \dots

Let $N = \{1, 2\}$ be the set of *agents*.

Let \mathcal{P} be the set of linear orders on \mathcal{A} ; linear orders are complete, transitive, antisymmetric binary relations. Elements in \mathcal{P} stand for the *preferences* of agents and are denoted by P_i ($i=1, 2$). For $B \subset \mathcal{A}$, $P \in \mathcal{P}$, $C(P, B) = \{x \in B \mid xPy \text{ for all } y \in B\}$. We denote by \mathcal{P}_x the set $\{P \in \mathcal{P} \mid C(P, \mathcal{A}) = x\}$, that is, the set of preferences whose best alternative is x .

Elements of \mathcal{P}^2 are called *preference profiles*. Profiles are denoted by (P_1, P_2) , (P'_1, P'_2) , etc., ..., or more compactly by \mathbf{P}, \mathbf{P}' . Where \mathbf{P} is a profile (P_1, P_2) and $P'_i \in \mathcal{P}$, \mathbf{P}/P'_i denotes the profile obtained from \mathbf{P} by changing P_i to P'_i .

Definition 2.1. A *voting scheme* is a function $f: \mathcal{P}^2 \rightarrow \mathcal{A}$. The range of f is the set of alternatives $r_f = \{x \in \mathcal{A} \mid \exists \mathbf{P} \in \mathcal{P}^2 \text{ such that } f(\mathbf{P}) = x\}$. $\#r_f$ denotes the cardinality of r_f .

Definition 2.2. A voting scheme is *manipulable* if there exists $\mathbf{P} \in \mathcal{P}^2$, $P'_i \in \mathcal{P}$ such that $f(\mathbf{P}/P'_i) \neq f(\mathbf{P})$ and $f(\mathbf{P}/P'_i) P_i f(\mathbf{P})$.

We then say that agent i can manipulate f at \mathbf{P} via P'_i .

Definition 2.3. A voting scheme is *dictatorial* if there exists $i \in N$ such that, for all $\mathbf{P} \in \mathcal{P}^2$, $f(\mathbf{P}) = C(P_i, r_f)$.

Theorem 2.4. (Gibbard-Satterthwaite). Let f be a voting scheme, with $\#r_f > 2$. Then f is either manipulable or dictatorial.

Proof of the Theorem. We assume that f is not manipulable and that $\#r_f > 2$, and prove that it must be dictatorial.

Define $O_2(P_1) = \{x \mid \exists P_2 \text{ such that } f(P_1, P_2) = x\}$. We call $O_2(P_1)$ the *set of options for agent 2, given P_1* . The proof follows from the following facts about sets of options for non manipulable voting schemes, which are presented as Lemmas.

Lemma 2.5. $(\forall P) f(P) = C(P_2, O_2(P_1))$.

Proof of Lemma 2.5. Let $z = f(P)$ and $x = C(P_2, O_2(P_1))$. By definition of $O_2(P_1)$, $\exists \bar{P}_2$ such that $f(P_1, \bar{P}_2) = x$. If $x \neq z$, xP_2z , and 2 can manipulate f at (P_1, P_2) via \bar{P}_2 .

Lemma 2.6. $(\forall P_1) C(P_1, r_f) \in O_2(P_1)$.

Proof of Lemma 2.6. Let $x = C(P_1, r_f)$. Since $x \in r_f$, $\exists \bar{P} = (\bar{P}_1, \bar{P}_2)$ such that $f(\bar{P}) = x$. If $f(P_1, \bar{P}_2) = z \neq x$, we have that xP_1z and 1 can manipulate f at (P_1, \bar{P}_2) via \bar{P}_1 .

Corollary to Lemmas 2.5 and 2.6

$$(\forall x \in r_f) [P_1, P_2 \in \mathcal{P}_x \rightarrow f(P_1, P_2) = x].$$

Lemma 2.7. $(\forall x \in r_f) [P_1, P'_1 \in \mathcal{P}_x \rightarrow O_2(P_1) = O_2(P'_1)]$.

Proof of Lemma 2.7. Suppose not. Without loss of generality, there will exist a $z \in O_2(P_1) - O_2(P'_1)$. By Lemma 2.6, $x \in O_2(P_1)$ and $x \in O_2(P'_1)$. Let \bar{P}_2 be such that $z\bar{P}_2x\bar{P}_2w$, for all $w \in r_f, w \notin \{z, x\}$. That is, \bar{P}_2 ranks z in first place and x in second among all alternatives in the range. Then, $f(P_1, \bar{P}_2) = z$, while $f(P'_1, \bar{P}_2) = x$, and 1 can manipulate f at (P_1, \bar{P}_2) via P'_1 , a contradiction.

Lemma 2.8. *For all $P_1 \in \mathcal{P}$ either $O_2(P_1)$ is a singleton or $O_2(P_1) = r_f$.*

Proof of Lemma 2.8. Suppose not. Then, there would exist $\bar{P}_1 \in \mathcal{P}$ and $x, y, z \in r_f$ such that $x, y \in O_2(\bar{P}_1), z \notin O_2(\bar{P}_1)$. We can assume, without loss of generality, that $\bar{P}_1 \in \mathcal{P}_x$ and that $z\bar{P}_1y$ (by Lemma 2.7). Consider now \bar{P}_2 such that $z\bar{P}_2y\bar{P}_2w$ for all $w \in r_f, w \notin \{z, y\}$. We now have $f(\bar{P}_1, \bar{P}_2) = y$. However, if $P'_1 \in \mathcal{P}_z$, we have $z \in O_2(P'_1)$ and thus $f(P'_1, \bar{P}_2) = z$. But then, 1 can manipulate f at (\bar{P}_1, \bar{P}_2) via P'_1 , a contradiction.

Lemma 2.9. *Either $O_2(P_1)$ is a singleton for all $P_1 \in \mathcal{P}$, or else $O_2(P_1) = r_f$ for all $P_1 \in \mathcal{P}$.*

Proof of Lemma 2.9. By Lemma 2.8, the only possibility for the statement not to hold is that $O_2(\bar{P}_1) = r_f$ for some \bar{P}_1 and $O_2(\hat{P}_1) = x$ for some \hat{P}_1, x . We can assume, by Lemma 2.7 and the fact that r_f consists of more than two alternatives, that $x\bar{P}_1z$, for some $z \in r_f$. Consider now any $P_2 \in \mathcal{P}_z$. We have that $f(\bar{P}_1, P_2) = z$, while $f(\hat{P}_1, P_2) = x$. Thus, 1 can manipulate f at (\bar{P}_1, P_2) via \hat{P}_1 , a contradiction.

To complete the proof that f is dictatorial, we must only remark that, if $O_2(P_1)$ is a singleton for all P_1 , it must be that $O_2(P_1) = C(P_1, r_f)$, by Lemma 2.6. Then, $f(P_1, P_2) = C(P_1, r_f)$ for all (P_1, P_2) and 1 is a dictator. While, if $O_2(P_1) = r_f$ for all P_1 , then $f(P_1, P_2) = C(P_2, r_f)$, for all (P_1, P_2) , by Lemma 2.5, and 2 is a dictator.

To close this section, let us comment on the aspects of the proof which are generalizable and those which must be modified. The passage from two to more agents involves a very simple induction argument and does not require any substantial change. Allowing for indifferences does not pose any problem, provided we do not exclude from the domain those preferences having a single best alternative. The constructs that do not generalize immediately when we pass to continuous preferences on infinite sets of alternatives are those which use the notion of a "best" or a "second best" element, like those in the proofs of Lemmas 2.7, 2.8, or 2.9. The reader will find a more elaborate argument in the proof of the general theorem.

3. The Model, and the Theorem

\mathcal{A} , the set of *alternatives*, is now a metric space. The metric on \mathcal{A} is denoted by d . U is the set of *continuous* real-valued functions on \mathcal{A} . $N = \{1, \dots, n\}$ is the set of *voters* or *players*. If $S \subseteq N$, $S \neq \emptyset$, then we denote by U^S the set of all functions from S to U . Intuitively, U^S is the set of all *utility profiles* for S . If $u = (u_1, \dots, u_n) \in U^N$, $i \in N$, and $u_i^1 \in U$ then $\hat{u} = u/u_i^1$ denotes the profile where $\hat{u}_j = u_j$ for all $j \neq i$, and $\hat{u}_i = u_i^1$.

A *voting scheme* (VS) is a function $f: U^N \rightarrow \mathcal{A}$. The range of f is denoted by r_f . $i \in N$ is a *dictator* for a VS f if for all $u \in U^N$ and for all $x \in r_f$, $u_i(f(u)) \geq u_i(x)$ (where, here and in the sequel, if $u \in U^N$ and $j \in N$ then u_j is the j -th component of u). A VS f is *dictatorial* if it has a dictator.

A VS is *manipulable* if there exist $u \in U^N$, $i \in N$, $u_i^1 \in U$ such that $u_i(f(u/u_i^1)) > u_i(f(u))$. We then say that i can manipulate f at u via u_i^1 . If f is not manipulable, it is *strategy-proof* (SP). We shall prove the following theorem.

Theorem 3.1. *A strategy proof voting scheme whose range contains at least three alternatives is dictatorial.*

Remark 3.2. If \mathcal{A} is a separable metric space, then every continuous preference order is representable by a continuous utility function. Therefore our theorem applies to voting schemes which are defined on continuous preorders, and thus on a subset of the domain for which the Gibbard-Satterthwaite theorem applies.

4. Preliminary Results

Let f be nonmanipulable. Let $(SM_f)^N$ be the subset of profiles where each agent's preferences have a single maximal element in r_f . Let f^* be the restriction of f to $(SM_f)^N$. In this section we prove that if f^* is dictatorial (on $(SM_f)^N$), then f is also dictatorial (on U^N). This will allow us to concentrate, later on, on the properties of f^* . Other useful results in this section prove that the range of f^* equals the range of f and is closed. A starting point for all these is the fact that, if f is strategy proof and all agents consider $x \in r_f$ to be their single best alternative then the outcome of f is x . All these facts are now formally stated and proved.

Notation 4.1. Let $x \in r_f$. We denote $T_x = \{u \in U | u(x) > u(y) \text{ for all } y \in r_f - \{x\}\}$.

Notation 4.2. $SM_f = \{w \in U \mid w \in T_x \text{ for some } x \in r_f\}$. Thus SM_f is the set of utility functions which have a unique global maximum on r_f .

Definition 4.3. Let $f^* = f|(SM_f)^N$. (Clearly f^* is SP (on $(SM_f)^N$)).

We first show that an SP VS satisfies a unanimity condition.

Lemma 4.4. *If f is an SP VS, $x \in r_f$, and $u_i \in T_x$ for $i=1, \dots, n$, then $f(u) = x$.*

Proof. Let $x = f(\hat{u}_1, \dots, \hat{u}_n)$. Assume, on the contrary that $f(u) \neq x$. Let

$$z_i = f(u_1, \dots, u_i, \hat{u}_{i+1}, \dots, \hat{u}_n), i=0, \dots, n.$$

Then $z_0 = x$ and $z_n \neq x$. Hence there exists $1 \leq j \leq n$ such that $z_{j-1} = x$ and $z_j \neq x$. Therefore, f is manipulable by j at $(u_1, \dots, u_j, \hat{u}_{j+1}, \dots, \hat{u}_n)$ via \hat{u}_j .

Henceforth we assume that f is an SP VS.

Corollary 4.5. $r_f = r_{f^*}$.

Definition 4.6. A voting scheme $f: U^N \rightarrow \mathcal{A}$ satisfies the *strong positive association property* (SPAP) if for every $u \in U^N, i \in N, u_i^1 \in U$, if $f(u) = x$ and $[u_i(y) \leq u_i(x) \Rightarrow u_i^1(y) \leq u_i^1(x)]$ for all $y \in r_f$, then $f(u/u_i^1) = x$.

We remark that an SP VS may not satisfy the SPAP. This is because indifference sets may be non-trivial. However, an SP VS satisfies a slightly modified version of SPAP.

Definition 4.7. A voting scheme $f: U^N \rightarrow \mathcal{A}$ satisfies the *modified SPAP* (MSPAP) if for every $u \in U^N, i \in N, u_i^1 \in U$, if $f(u) = x$ and

$$[u_i(x) \geq u_i(y) \text{ and } y \neq x \Rightarrow u_i^1(x) > u_i^1(y), \text{ for all } y \in r_f],$$

then $f(u/u_i^1) = x$.

Lemma 4.8. *If a VS f is SP, then it satisfies the MSPAP.*

Proof. Let $u \in U^N, f(u) = x, i \in N, u_i^1 \in U$ and let $[u_i(x) \geq u_i(y) \text{ and } y \neq x \Rightarrow u_i^1(x) > u_i^1(y), \text{ for all } y \in r_f]$. Assume, on the contrary, that $f(u/u_i^1) = z$ and $z \neq x$. If $u_i^1(z) \geq u_i^1(x)$ then $u_i(z) > u_i(x)$ and f is manipulable at u by i via u_i^1 . On the other hand, if $u_i^1(z) < u_i^1(x)$, then f is manipulable at u/u_i^1 by i via u_i .

Lemma 4.9. *If $x, y \in r_f, u \in U^N$, and $u_i(x) > u_i(y)$ for all $i \in N$, then $f(u) \neq y$.*

Proof. Assume on the contrary, that $f(u) = y$. Choose $\hat{u}_i (\forall i \in N)$ so that (i) $\hat{u}_i \in T_x$ and (ii) $u_i(y) \geq u_i(z)$ and $z \neq y$ imply that $\hat{u}_i(y) > \hat{u}_i(z)$. Then $f(\hat{u}) = y$ by MSPAP, while Lemma 4.4 would require $f(\hat{u}) = x$, a contradiction.

Corollary 4.10. r_f is closed.

Proof. Suppose that x is in the closure of r_f . Define u by setting $u_1(y) = -d(x, y) \forall y \in \mathcal{A}$, and $u_i = u_1 \forall i \in N$. If $y \neq x$, there is some $x' \in r_f$ such that $d(x, x') < d(x, y)$. Then $u_i(x') > u_i(y)$ for all $i \in N$, and thus $f(u) \neq y$ by Lemma 4.9. Then $f(u) = x$.

Lemma 4.11. *If f^* is dictatorial (on $(SM_f)^N$), then f is dictatorial (on U^N).*

Proof. Let i be a dictator for f^* . We shall prove that i is a dictator for f . Let $u \in U^N$ and let $f(u) = x$. Assume, on the contrary, that there exists $y \in r_f$ such that $u_i(y) > u_i(x)$. Let $\hat{u}_j \in T_x$ for $j \neq i$, and let $\hat{u}_i \in T_y$ satisfy

$$u_i(x) \geq u_i(z) \quad \text{and} \quad z \neq x \Rightarrow \hat{u}_i(x) > \hat{u}_i(z) \quad \text{for all } z \in r_f.$$

Then $\hat{u} \in (SM_f)^N$ and by MSPAP $f(\hat{u}) = f^*(\hat{u}) = x$. Now $\hat{u}_i(y) > \hat{u}_i(x)$ and $y \in r_{f^*}$ which contradicts our assumption that i is a dictator for f^* .

5. A Proof of the Theorem

Let f be SP. We want to prove that it is dictatorial. Given the results in Sect. 4, it will suffice to prove that f 's restriction on $(SM_f)^N$, f^* , is dictatorial. Obviously, we can use the fact that f^* is strategy proof (on $(SM_f)^N$), if f is strategy proof (on U^N).

As in the simple case in Sect. 2, our proof uses the properties of option sets $O_{-i}(u_i)$.

Notation 5.1. If $i \in N$ then members $u^{N-(i)} \in U^{N-(i)}$ will be denoted by u_{-i} .

Definition 5.2. Let $u_i \in SM_f$. We define

$$O_{-i}(u_i) = \{x | x = f(u_i, u_{-i}), u_{-i} \in (SM_f)^{N-(i)}\}$$

Lemma 5.3. $O_{-i}(u_i)$ is closed, for all $u_i \in SM_f$.

Proof. It is a range of an SP VS $g(u_{-i}) = f(u_i, u_{-i})$ (see Corollary 4.10).

Notation 5.4. If $x \in \mathcal{A}$ and $\delta > 0$

$$B(x, \delta) = \{y \in \mathcal{A} | d(x, y) < \delta\}.$$

Lemma 5.5. For all u such that $u_j = u_k$ for $j, k \neq i$,

$$f(u) = \operatorname{argmax}_{t \in O_{-i}(u_i)} u_j(t)$$

Proof. Follows from the fact that the function $g(u_{-i}) = f(u_i, u_{-i})$ is strategy proof and satisfies Pareto Optimality by Lemma 4.9.

Lemma 5.6. If $x \in r_f$ and $\bar{u}_i, \bar{u}_i \in T_x$, then $O_{-i}(\bar{u}_i) = O_{-i}(\bar{u}_i)$.

Proof. Assume, on the contrary, that there exists, say, $z \in O_{-i}(\bar{u}_i) - O_{-i}(\bar{u}_i)$. By Lemma 4.4 $x \in O_{-i}(\bar{u}_i)$. Hence $z \neq x$. Let $\delta > 0$ satisfy $B(z, 2\delta) \cap O_{-i}(\bar{u}_i) = \emptyset$. Define

$$\hat{u}_j(t) = \frac{d(t, \mathcal{A} - B(z, \delta))}{d(t, z) + d(t, \mathcal{A} - B(z, \delta))}, \quad \forall j \neq i. \quad \text{Then } \hat{u}_j \in T_z \text{ and } \hat{u}_j(t) = 0$$

for $t \in O_{-i}(\bar{u}_i)$. Now define

$$u_j^*(t) = \frac{d(t, \mathcal{A} - B(x, \delta))}{2(d(t, x) + d(t, \mathcal{A} - B(x, \delta)))}, \quad \forall j \neq i. \quad \text{Notice that } B(x, \delta) \cap B(z, \delta) = \emptyset.$$

Now let $v_j = \hat{u}_j + u_j^*$, $j \neq i$. Consider first the function $\bar{g}(u_{-i}) = f(\bar{u}_i, u_{-i})$; $\bar{g}(v_{-i})$. It is SP with range $O_{-i}(\bar{u}_i)$. By Lemma 5.5., $\bar{g}(v_{-i}) = z = f(\bar{u}_i, v_{-i})$ since $z \in O_{-i}(\bar{u}_i)$ and $v_j(z) > v_j(x')$ for all $x' \in O_{-i}(\bar{u}_i) - \{z\}$ and $j \neq i$. Now consider $\bar{g}(u_{-i}) = f(\bar{u}_i, u_{-i})$. Again by Lemma 5.5 and a similar reasoning, $\bar{g}(v_{-i}) = x = f(\bar{u}_i, v_{-i})$. Hence f is manipulable by i at (\bar{u}_i, v_{-i}) via \bar{u}_i , which is the desired contradiction.

Lemma 5.7. *Either*

- (a) $O_{-1}(u_1)$ is a singleton for each $u_1 \in SM_f$, or
 (b) $O_{-1}(u_1) = r_f$ for all $u_1 \in SM_f$.

Proof. Suppose not. Then there exists \bar{u}_1 and distinct $x, y, z \in r_f$, such that

$$x, y \in O_{-1}(\bar{u}_1), z \notin O_{-1}(\bar{u}_1).$$

Assume, w.l.o.g. that $\bar{u}_1 \in T_x$ and (by Lemma 5.6.), that $\bar{u}_1(z) > \bar{u}_1(y)$.

Let \hat{u}_2 be a function in T_z such that $\operatorname{argmax}_{t \in O_{-1}(\hat{u}_1)} \hat{u}_2(t) = y$.

This construction is possible by Lemma 5.3. By Lemma 5.5, $f(\bar{u}_1, \bar{u}_{-1}) = y$, when $\hat{u}_j = \bar{u}_j$ for all $j \neq 1$. By Lemma 4.4, $f(\hat{u}_1, \hat{u}_{-1}) = z$, when $\hat{u}_1 = \hat{u}_2$. Thus, 1 can manipulate at $(\bar{u}_1, \bar{u}_{-1})$ via \hat{u}_1 .

Proof of Theorem 3.1

The proof is by induction on n , the number of voters. The case $n=1$ is straightforward. Suppose the theorem holds for all $k=1, \dots, n-1$, and let f be defined for n agents.

If (a) of Lemma 5.7 holds, agent 1 is a dictator, since the singleton $O_{-1}(u_1)$ must always be the maximal element on r_f of u_1 .

If (b) holds, $g_{u_1}(u_{-1}) \equiv f(u_1, u_{-1})$ is an $n-1$ SP VS for any u_1 , and thus dictatorial by the induction hypothesis.

If the dictator is the same agent for all u_1 , the proof is complete. Suppose not, i.e., $\exists \bar{u}_1, \hat{u}_1, i, j (i \neq j)$ such that i is a dictator for $g_{\bar{u}_1}(\cdot)$ and j is a dictator for $g_{\hat{u}_1}(\cdot)$. Because $\bar{u}_1 \in SM_f$, there exist $x, y \in r_f$ such that (w.l.o.g) $\bar{u}_1(x) > \bar{u}_1(y)$. Consider a profile $u = (\bar{u}_1, \bar{u}_{-1})$ such that $\bar{u}_i \in T_y$ and $\bar{u}_j \in T_x$. For this profile, $f(\bar{u}_1, \bar{u}_{-1}) = y$, since y is i 's best alternative and i is a dictator for $g_{\bar{u}_1}(\cdot)$. But $f(\hat{u}_1, \bar{u}_{-1}) = x$, since j is now a dictator for $g_{\hat{u}_1}(\cdot)$. Thus, agent 1 could manipulate at $(\bar{u}_1, \bar{u}_{-1})$ via \hat{u}_1 , a contradiction. Therefore, the function f is dictatorial, and this completes the proof of the theorem.

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